ON THE UNCERTAINTY PRINCIPLE: THE STATISTICAL APPROACH

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1 — INTRODUCTION

The uncertainty principle is usually illustrated, at the freshman level, with the "Heisenberg microscope experiment" for the position and linear momentum *simultaneous determination* of a microphysical particle. From this experiment emerges the general idea that the interaction between the *observed system* and the *experimental apparatus* prevents the exact and simultaneous measurement of the position and momentum from being made. The *link* between the observed system and the apparatus in the Heisenberg microscope — the colliding photon being diffracted by the objective of the microscope and causing an uncertainty in the position determination, Δx — causes a Compton effect alteration of the linear momentum, Δp_x, related with Δx by the order of magnitude relationship

$$\Delta x \, \Delta p_x \simeq h. \tag{1}$$

At a more advanced level, a precise expression of the Heisenberg principle,

$$\Delta x \, \Delta p_x \geq h/4\pi \tag{2}$$

is derived by the application of statistical arguments to a Schwarz type inequality which thus yields a general expression for the derivation of uncertainty relationships [1],

$$\Delta A \;\Delta B \geq (1/2) \mid \int \psi^*[\hat{A},\hat{B}]\psi d\tau \mid \tag{3}$$

where ΔA and ΔB, the so-called uncertainties in A and B, are the positive square roots of the variances of A and B,

$$(\Delta A)^2 = \int \psi^*(\hat{A} - <A>)^2 \psi d\tau, \tag{4a}$$

$$(\Delta B)^2 = \int \psi^*(\hat{B} - <B>)^2 \psi d\tau. \tag{4b}$$

It would seem at first sight possible to evaluate the standard deviatons, ΔA and ΔB, without any interfering connection whatsoever between them, all the more since ΔA and ΔB are supposed to be evaluated by separate measurements being performed in totally different subensembles of microphysical particles. However, it is not immediately apparent where does the requirement of *simultaneous measurements* exhibited by the microphysical approach has its equivalent counterpart in the statistical approach. By just mentioning the Schwarz

type inequality as the basic mathematical relationship used in the derivation of (3) one does not necessarily explain it.

While several implications of the uncertainty principle have been previously considered [2,3] an appreciation of the statistical approach insofar as the simultaneity of measurements is concerned has so far been ignored thus prompting us to assess it in this article. It is concluded that the simultaneity of measurements which naturally emerges from microphysical experiments used in the illustration of the uncertainty principle is clearly implemented in the more natural and less enigmatic statistical approach.

2 — THE STATISTICAL APPROACH AND THE CONCEPT OF SIMULTANEOUS MEASUREMENTS

If two observables A and B have to be compatible, *i.e., simultaneously assignable* to the system for *all* eigenvalues, then they must have a *common complete set* of eigenfunctions. A necessary and sufficient condition for two observables to be simultaneously measurable with precision is that they *commute* [1]. Therefore, if $\hat{A}$ and $\hat{B}$ are two Hermitian operators which do not commute, the physical quantities A and B cannot both be sharply defined simultaneously and the degree to which an inevitable lack of precision in A and B has to be admitted is measured by their commutator, $[\hat{A},\hat{B}]$, which is an anti-Hermitian operator, *i.e.*,

$$[\hat{A},\hat{B}]=i\hat{C} \qquad (5)$$

where $\hat{C}$ is a Hermitian operator [1].

To the extent to which the Heisenberg uncertainty relationship is a direct consequence of the noncommutivity of the position and momentum operators, it can be said that the *concept of simultaneous measurements* of A and B clearly emerging from the microphysical experiment, is implemented in the quantum mechanical formalism by comparing, in the right hand side of (3), the results of applying $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$ to the *same state function* ψ. This seems to be a generally understood feature of the quantum mechanical formalism. In fact, the measurement of an observable A is formally represented by applying the corresponding Hermitian operator to the state function. Hence, if the comparison of consecutive measurements in reversed orders, $\hat{A}\hat{B}$ and $\hat{B}\hat{A}$, does not lead to the same result, one cannot claim simultaneous measurability of both observables. In this way one can qualitatively appreciate the physical meaning of the lower limit established by (3). It should be mentioned that the simultaneity of measurements as it is thus translated in the quantum mechanical formalism is much more restrictive than one might at first think. In fact, if two operators are applied in sequence, the second is operating on a function which is not the original state function and might well be outside the domain where the second operator is Hermitian, *i.e.*, where it represents a physical quantity [4,5]. For example, when $\hat{A}$ is applied after $\hat{B}$ (operator product sequence $\hat{A}\hat{B}$), $\hat{A}$ is in fact applied to $\hat{B}\psi$ which must be a function of the A-Hermitian operator domain. Of course, the same reasoning should also refer to $\hat{B}$ applied after $\hat{A}$ and the functions $\hat{A}\psi$ (operator product sequence $\hat{B}\hat{A}$) are likewise required to belong to the domain where $\hat{B}$ is a Hermitian operator. If these restrictive conditions do not hold, then one easily arrives at paradoxal results [4,5] like $<\psi \mid [\hat{A},\hat{B}] \mid \psi>=0$, whenever ψ represents an eigenfunction of $\hat{A}$ or $\hat{B}$.

Considering the purely mathematical nature of (3), one can still argue that its experimental confirmation requires no more than the evaluation of its left hand side, *i.e.*, the statistical determination of ΔA and ΔB uncertainties. These are *independently* obtained by the statistical analysis of *separate subensembles* of microphysical systems. Along these lines, the evaluation of a large number of $\Delta A.\Delta B$ products would certainly confirm the validity of (3) by never going beyond the lower limit it establishes. Therefore, it is not immediately clear how *apparently unrelated experiments* for ΔA and ΔB evaluation could ever provide a lower limit to their product unless it was by clumsiness of instruments or by human inability. The answer to this question is provided by the fact that the ensemble has to be in the *same state* whenever A and B uncertainties are experimentally determined. The common state is the *link* between A and B measurements in the statistical approach. Therefore, the "unrelated way" of obtaining ΔA and ΔB is only apparent. This reasoning enables us to stress the fundamental nature of these uncertainties as opposed to the wrong idea of clumsiness of instruments which often easily emerges from microphysical experiments and leaves the student

in a confused state of mind [2]. Moreover, while the determination of the Heisenberg uncertainties by the statistical approach is in perfect agreement with the concept of statistical and macroscopic determinism, the microphysical approach often raises doubts as to its eventual conflict with the microphysical indeterminism.

3 — CONCLUSION

The simultaneity of measurements which naturally emerges from microphysical experiments used in the illustration of the uncertainty principle is clearly implemented in the more natural and less enigmatic statistical approach

i) by requiring the *same state* for the sub-ensembles where A and B measurements are performed;

ii) by comparing, in the quantum mechanical formalism, the results of consecutive measurements of A and B considered in reversed orders and

iii) by restricting the functions representing the state of the ensemble where A and B measurements are performed to *physically acceptable* situations [5] as it is required by the fact that the commutador (5) on the right hand side of (3) should be treated as a single operator.

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ABSTRACT

In this note the author shows how the statistical approach of the uncertainty principle accounts for the concept of simultaneity of measurements. This simultaneity is inherent to the microphysical experiments so often used to illustrate the uncertainty principle, like the "Heisenberg microscope experiment".

RESUMO

Nesta nota o autor mostra como é que a interpretação estatística do princípio da incerteza dá conta da noção de simultaneidade de medidas. Esta simultaneidade é inerente às experiências microfísicas tão vulgarmente usadas para ilustrar o princípio da incerteza, como a "experiência do microscópio de Heisenberg".